In this paper, we use Japanese government bond yield curve data to estimate the shadow rate from a shadow rate term structure model. We firstly confirm the traceability of shadow rate as a consistent and compatible measure of monetary policy which can be used to gauge the stance of monetary policy implemented by Bank of Japan, then we introduce shadow rate in a standard New Keynesian DSGE model and estimate the model using shadow rate estimated previously. Finally, we check the empirical results from the New Keynesian model. Empirical results show a good compatibility of the shadow rate used as a proxy of monetary policy in the estimation of New Keynesian model, providing credible policy implications as the benchmark New Keynesian model does in a standard context without the restriction of zero lower bound. Also, it has been shown that the monetary easing policy conducted by Bank of Japan has significantly positive effects on the improvement of the output gap and deflation. Using the shadow rate in the estimation of DSGE models also avoids the technical difficulties incurred by the nonlinearity of the zero lower bound.

Key Words: shadow rate, New Keynesian model, unconventional monetary policy, Bank of Japan, zero lower bound

JEL classification: E32, E44, E52

INTRODUCTION

When the short-term nominal interest rate is at or near zero, central banks have to face the problems incurred by the Zero Lower Bound (ZLB) because the ZLB invalidates the implementation of conventional monetary policy, the adjustment of short-term policy rate. Facing the constraint of ZLB, central banks conduct unconventional monetary policy to stabilize and stimulate the economy. This is what Japan economy has experienced and Bank of Japan (BoJ) has done since 1999 when the call rate decreased to a very low level near zero. The Great Recession incurred by the Global Financial Crisis 2007-2009 brought the same problem to US, UK and Euro area. Besides the practical policy issues faced by the monetary authorities in advanced economies, the ZLB and the related unconventional monetary policy also pose academic issues and new challenges for macroeconomic research.

New Keynesian Dynamic Stochastic General Equilibrium (NK-DSGE) model and Gaussian Affine Term Structure (GATS) model are two main workhorses in modern monetary macroeconomics and macro-finance. But when the ZLB on the short-term nominal interest rate binds, unfortunately, these models have deficient performance and undesired economic implications, leading to implausible and weird policy paradoxes and unsatisfied fitness of data. As a standard methodology for monetary policy analysis, a NK-DSGE model in the ZLB environment predicts that positive temporary supply from supply side of economy have contractionary effects and vice versa, negative shocks from supply side of economy have expansionary effects. Also, fiscal and forward guidance multipliers can be implausibly larger than one. But empirical works such as Wieland (2015) and Garín et al. (2019) showed similar impulse responses of output to a supply shock in both ZLB and non-ZLB environment. All these conclusions from the standard NK-DSGE models are inconsistent with economic intuition and empirical facts. Besides the misleading policy implications, the ZLB also
brings many technical problems in DSGE methodology. The explicit introduction of the ZLB constraint into DSGE models accompanies with structural break or nonlinear kink. Such kind of nonlinearity invalidates the linear approximation and the Kalman filter. Some researchers use global projection method and the particle filter to deal with the nonlinearity in solution and estimation of DSGE models, but these methods are technically difficult and demand for numerous computations. As another main methodology in macro-finance research, GATS models which are widely used to fit the term structure of interest rate can provide good description of the dynamics of short interest rate in many macro-financial applications in the non-ZLB environment, but in the ZLB environment, the GATS models provide barely satisfactory fitness of data and cannot accommodate the "sticky" property of short interest rate. This "sticky" property means that the short-term interest rates tend to keep approximately static around the ZLB with lower volatilities for a long period of time.

LITERATURE REVIEW

In recent research, the Shadow Rate Term Structure (SRTS) model which was firstly proposed by Black (1995) has become an alternative to overcome the poor performance incurred by the ZLB in general GATS model. Kim and Singleton (2012) and Bauer and Rudebusch (2013) have used the shadow rate estimated from a SRTS model to capture the behavior of interest rates and the stance of unconventional monetary policy in the ZLB environment. Krippner (2013) proposed a continuous-time formulation of SRTS model, known as Krippner Affine Gaussian model (K-AGM), where he added a call option feature to derive the closed-form solution. Wu and Xia (2016) took an analogous discrete-time approach which can be directly applied to discrete-time yield curve data without any numerical error associated with simulation methods and numerical integration in other models. All of these literatures advocate the effectiveness of the shadow rate as a kind of measure for describing the monetary policy stance.

Wu and Zhang (2016) established the equivalence between shadow rate and unconventional monetary policy in a standard NK-DSGE model. The equivalence between shadow rate and unconventional monetary policy is established on the empirical findings that have shown the highly correlation between the quantity of government bond purchase and the estimated shadow rate. The shadow rate can take both positive and negative values and show consistent response to monetary policy in both non-ZLB and ZLB environment. Introducing the shadow rate into a DSGE model can provide more insights for the propagation and amplification mechanism of unconventional monetary policy without introducing the complications incurred by the ZLB constraint.

Methodology, Hypothesis and Main Conclusions

We take a combination of these two methods, using the shadow rate estimated from a SRTS model as the data for the estimation of a NK-DSGE model where the general policy rate is replaced by the shadow rate. Then we use the NK-DSGE model with shadow rate to do some monetary policy analysis for Japan economy. This may be a circuitous route, but the logic is valid and consistent from the beginning to the end. Also, this approach salvages the DSGE models from the nonlinearity incurred by the ZLB. Standard procedures such as the linear approximation and the Kalman filter can be used instead of complicated nonlinear solution and estimation techniques. The main hypothesis in this paper is that we can use the shadow rate as a proxy of monetary policy in the monetary policy analysis with DSGE models without considering the technical difficulties of zero lower bound explicitly. Main conclusion from this paper is that we can still use general linear solution and estimation techniques of DSGE models in the monetary policy analysis by introducing the concept of shadow rate with a semi-structural approach proposed by Wu and Zhang (2016). Estimation results from full-sample and sub-sample shows that the structural parameters are quite robust in the Bayesian estimation with the estimated shadow rate. Also, historical decomposition and impulse response also advocate the effectiveness of the unconventional monetary policy conducted by the BoJ.

The remaining of this paper is organized as follows. Section 1 presents the estimation of shadow rate from a SRTS model of Krippner (2013). We also check the shadow rate's traceability as a summary for monetary policy. In Section 2, we estimate a New Keynesian DSGE model with shadow rate. Section 3 checks the empirical results such as historical decomposition and impulse response from the estimated DSGE model. Section 4 concludes this paper and gives the prospect for further research.

1. Shadow Rate

Bauer and Rudebusch (2013) and Christensen and Rudebusch (2014) pointed out that the estimated shadow rates vary across different model specifications and estimation methods. Kim and Singleton (2012) and Bauer and Rudebusch (2013) use simulation-based estimation method to estimate the shadow rate. Krippner (2013) and Wu and Xia (2016) estimated an analogous SRTS model, but with different time specifications, the former is in continuous-time and the the latter is in discrete-time. Krippner (2013) and Wu and Xia (2016) pose their estimation codes on Internet. By replicating the estimation of shadow rates for US, UK, Japan and EU, the results don't show much difference in two methods. Ichiiye and Ueno (2013)'s estimation is based on the approximation of bond prices by ignoring Jensen's inequality. Imakubo and Nakajima (2015) also estimated a shadow rate model to
extract inflation risk premium from nominal and real term structures. By surveying related literature and replicating the estimation results, we find that although the shadow rates estimated from different model specifications and estimation methods do have different values, they show common trend and dynamics and lead to same economic implications in the analysis of monetary policy, at least for the major advanced economies, US, UK, Japan and Euro area.

Also, most existing literatures advocate the potential usefulness of shadow rate as a measure for the stance of monetary policy. Bullard (2012), Krippner (2012), Lombardi and Zhu (2014) and Wu and Xia (2016) all support the view that the shadow rate is a powerful tool to summarize useful information from yield curve data and describe the conventional monetary policy stance in the non-ZLB environment and unconventional monetary policy stance in the ZLB environment in a consistent manner.

1.1. Estimation of Shadow Rate

Krippner (2012, 2013, 2014, 2015a, 2015b) provide a detailed introduction of his methodology of term structure modeling of shadow rate. For derivation of the model, please refer to Krippner (2015b). We use yield curve data of Japanese government bond to estimate the shadow rate. See Table 1 for the summary statistics of yield curve data.

Table 1: Summary Statistics of Japan Yield Curve Data

<table>
<thead>
<tr>
<th>Maturity</th>
<th>3M</th>
<th>6M</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
</tr>
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<tbody>
<tr>
<td>Mean</td>
<td>0.46</td>
<td>0.47</td>
<td>0.51</td>
<td>0.63</td>
<td>0.77</td>
<td>0.94</td>
</tr>
<tr>
<td>Med</td>
<td>0.12</td>
<td>0.13</td>
<td>0.15</td>
<td>0.25</td>
<td>0.40</td>
<td>0.57</td>
</tr>
<tr>
<td>Max</td>
<td>4.29</td>
<td>4.17</td>
<td>4.16</td>
<td>4.24</td>
<td>4.41</td>
<td>4.95</td>
</tr>
<tr>
<td>Min</td>
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<td>-0.43</td>
<td>-0.37</td>
<td>-0.36</td>
<td>-0.36</td>
<td>-0.36</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>0.85</td>
<td>0.83</td>
<td>0.85</td>
<td>0.90</td>
<td>0.97</td>
<td>1.06</td>
</tr>
<tr>
<td>Obs.</td>
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<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Maturity</th>
<th>5Y</th>
<th>7Y</th>
<th>10Y</th>
<th>15Y</th>
<th>20Y</th>
<th>30Y</th>
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<tbody>
<tr>
<td>Mean</td>
<td>1.10</td>
<td>1.40</td>
<td>1.81</td>
<td>2.13</td>
<td>2.53</td>
<td>2.70</td>
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<tr>
<td>Med</td>
<td>0.73</td>
<td>1.03</td>
<td>1.49</td>
<td>1.83</td>
<td>2.20</td>
<td>2.48</td>
</tr>
<tr>
<td>Max</td>
<td>5.25</td>
<td>5.71</td>
<td>5.64</td>
<td>6.16</td>
<td>6.44</td>
<td>6.24</td>
</tr>
<tr>
<td>Min</td>
<td>-0.37</td>
<td>-0.39</td>
<td>-0.28</td>
<td>-0.13</td>
<td>0.03</td>
<td>0.05</td>
</tr>
<tr>
<td>Std. Dev</td>
<td>1.13</td>
<td>1.22</td>
<td>1.26</td>
<td>1.29</td>
<td>1.26</td>
<td>1.16</td>
</tr>
<tr>
<td>Obs.</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
<td>6360</td>
</tr>
</tbody>
</table>

The Figure 1 shows the plot of data from which we can find that since 1999, the yield curve has shifted down to very low level. The 3-months bond yield has begun to take negative values since 2014M10. The yield curve data of Japan is Japanese government bond data from 1992-Jul-10 to 2016-Nov-24, daily frequency of 5-business days week with 6360 observations, obtained from the Bloomberg database. The maturity of yield curve is from 3-months to 30-years and 3-months interest rate is adapted to be the short interest rate, which is almost same as the short policy rate of BoJ. Interest rates with other maturities can be represented as a function of short interest rate and forward rate in term structure models. The Figure 2 gives the estimation result of shadow rate. The full series of daily estimation for shadow rate from 1992-Jul-10 is available upon request. We did the daily-frequency estimation of the shadow rate and aggregated the daily-frequency data to average monthly-frequency data. The gray shaded band around the point estimate (red line in Figure 2) is the 95% of the confidence interval. During non-ZLB period, the shadow rate and call rate have almost same dynamics. The shadow rate shows a good approximation to the call rate. But during the ZLB period, the call rate is approximately near to zero. We can’t read too much information about monetary policy from the call rate. But the shadow rate has keeping decreasing. In next section, we check the relation between the shadow rate and the balance sheet of BoJ. Besides other unconventional operations of the central bank, such as forward guidance and direct facility lending, the enlargement of balance sheet, widely known as quantitative easing (QE), is the essence of unconventional monetary policy, especially for the monetary policy of BoJ.

1.2. Empirical Evidence of Shadow Rate

We establish some empirical relations between the shadow rate and the balance sheet.
BoJ is the first central bank which introduced unconventional monetary policy among major advanced economies. Table 2 summarizes the monetary policy regimes of BoJ since 1999. The essence of the policy programs conducted by BoJ is the large scale purchase of Japanese government bond. From the Figure 3, we can find since 2010M10, the balance sheet of BoJ has increased aggressively. What is the relation between the shadow rate and the size of balance sheet?

Table 2: Monetary Policy Regimes of BoJ

<table>
<thead>
<tr>
<th>Year/Regime</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1999/2/2-2000/8/11: Zero Interest Rate Policy</td>
<td>(pink shaded area in Figure 4)</td>
</tr>
<tr>
<td>2001/3/19-2006/3/9: Quantitative Easing (yellow shaded area in Figure 4)</td>
<td></td>
</tr>
<tr>
<td>2010/10/5-2013/3/20: Comprehensive Monetary Easing (green shaded area)</td>
<td></td>
</tr>
<tr>
<td>2013/4/2016/1/29: Price Stability Target of 2% and QE</td>
<td>(blue shaded area in Figure 4)</td>
</tr>
<tr>
<td>2013/2-2016/9/19: Price Stability Target of 2% and QE</td>
<td></td>
</tr>
<tr>
<td>2016/9/21- Now: Price Stability Target of 2% and QE</td>
<td></td>
</tr>
</tbody>
</table>

The Figure 4 shows the time series plot of shadow rate, minus log of bond holdings and minus log of monetary base. The correlation between shadow rate and bond holdings is -0.81 and the correlation between shadow rate and monetary base is -0.82. The X-Y scatter plots also show obviously negative correlation between the shadow rate and the variables of balance sheet. Note that the shadow rate can’t be controlled directly by the central bank in the ZLB environment. In the non-ZLB environment, the shadow rate takes positive value which is same to the general short-term policy rate. The short-term policy rate can be controlled by the central bank through open market operations. What the central bank can manipulate is its balance sheet. The shadow rate just summarizes the monetary base and the variables of balance sheet. Note that the shadow rate can be controlled by the central bank through open market operations.

2. Shadow Rate NK-DSGE Model

In this section, we first derive a standard NK-DSGE model, then we incorporate the shadow rate into this standard framework. We show that the shadow rate can be used as an equivalence for QE so we can use the NK-DSGE model with shadow rate to analyze the unconventional monetary policy without considering the technical complexity incurred by the ZLB.

2.1. Standard NK-DSGE Model

We assume that representative household maximizes the discounted life-time utility $E_t \sum_{t=0}^{\infty} \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{x_t^{1+\eta}}{1+\eta}$ subject to budget constraint $c_t + \frac{b_t}{p_t} = \frac{R_t}{p_t} + W_t L_t + T_t$. Where $c_t$, $L_t$, $B_t$, $W_t$, $T_t$, $R_t$ represent consumption, labor, bond, real wage, lump-sum transfer and private interest rate respectively. $\sigma$ is the inverse of inter-temporal substitution of consumption and $\eta$ is the inverse of Frisch elasticity of labor supply. The representative household discounts the utility with objective discount factor $\beta$. Final-good firms produce homogenous final-goods with CES-type production function $Y_t = \left( \int_0^1 Y_{f,t} \frac{\varepsilon}{\varepsilon-1} d\xi \right)^{\frac{\varepsilon}{\varepsilon-1}}$ where $Y_{f,t}$ is an intermediate-good produced by intermediate-good firm. $\varepsilon$ is the elasticity of substitution among differentiated intermediate-goods. Intermediate-good firms use Cobb-Douglas production technology $Y_{f,t} = \alpha x_t^{1-\alpha}$ to produce heterogenous intermediate-goods where $\alpha$ is total factor productivity (TFP) that is common to all intermediate-good firms. As a basic setup in standard NK-DSGE model, sticky price is introduced by Calvo (1983)’s staggered price setting mechanism. For the details of derivation, please refer to Galí (2015), Walsh (2017) and Miao (2014). After solving the optimization problems of each economic agent, finally, the standard NK-DSGE model can be represented by two equations, New Keynesian IS equation describing the dynamics of output gap $x_t$ which can be derived from the inter-temporal optimization of household.
\[ x_t = E_t x_{t+1} - \frac{1}{\sigma} (r_t^B - E_t \pi_{t+1} - r_t^N) \]  \hspace{1cm} (1)

and the New Keynesian Phillips Curve (NKPC) equation describing the dynamics of inflation \( \pi_t \) which can be derived from the Calvo (1983)’s price setting mechanism.

\[ \pi_t = \beta E_t \pi_{t+1} + \kappa x_t \]  \hspace{1cm} (2)

Output gap \( x_t \) is defined as the difference between output under sticky price decentralized equilibrium and output under flexible price decentralized equilibrium. \( \kappa = \frac{1/(1-\alpha) + \alpha}{\theta(1-\alpha \cdot \alpha)} \) is a composite of deep structural parameters. We introduce the economic explanations of these deep structural parameters in next section. \( r_t^N \) is Natural interest rate which is determined by exogenous TFP process \( A_t \) in the standard context of New Keynesian mode.

### 2.2. Shadow Rate in NK-DSGE Model

According to the empirical evidence of shadow rate presented in Section 2, we introduce shadow rate into standard NK-DSGE model. For general NK-DSGE model, the economic agents face risk-free short rate \( r_t \) and hold risk-free bond. \( r_t \) is generally recognized as the short policy rate which can be controlled by the central bank. In actual, the relevant interest rates affecting economic agents’ decisions are private interest rates \( r_t^p \) through which both conventional and unconventional monetary policies transmit into the economy. Generally, the private interest rates \( r_t^p \) can be represented as the sum of risk-free short rate \( r_t \) plus a time-varying risk premium \( r_t^p = r_t + r_t^p \), where \( r_t \) is assumed that can be adjusted by the conventional monetary policy of the central bank. Empirical works such as Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2012) and Hamilton and Wu (2012) advocate that the large-scale asset purchase by the central banks can reduce the risk premium which means \( \frac{\partial r_t^p}{\partial t} < 0 \) where \( b_t^L \) is the log of bond holdings of the central bank. This is known as the risk premium channel of QE.

The Figure 5 shows the relation between credit spread and the balance sheet. The credit spread used here is defined as the difference between 10-years and 2-years Japanese government bond returns. The correlation between credit spread and log of bond holdings is -0.80 and the correlation of credit spread and log of monetary base is -0.81. According to the regression lines in Figure 5, we assume that the response of risk premium \( r_t^p \) to bond holdings \( b_t^L \) follows a simple linear form \( r_t^p = r^p - \gamma (b_t^L - b_0^L) + \epsilon_t^L \) where \( -\gamma = \frac{\partial r_t^p}{\partial b_t^L} < 0 \) , \( r^p \) is the constant component of risk premium and \( \epsilon_t^L \) is the exogenous time-varying component of risk premium which is interpreted as the liquidity preference shock in Campbell et al. (2017). In the non-ZLB environment, \( b_t^L = b_t^G \), \( r_t^p = r_t^G + \epsilon_t^L \) such that \( r_t^p = r_t + r_t^p = r_t + r_t^p + \epsilon_t^L \) which means that the private interest rate is the short rate controlled by the central bank plus risk premium. When \( r_t \) is restricted by the ZLB, approximately \( r_t = 0 \) and \( r_t^p = r_t^G = r_t^p + \epsilon_t^L \) through which the unconventional monetary policy affects risk premium to reduce private interest rate and stimulate the economy. According to the empirical evidence of the shadow rate, we also assume that the shadow rate has a same response to the log of bond holdings in a linear form like \( s_t = -\gamma (b_t^L - b_0^L) \), then \( r_t^p = s_t + r^p + \epsilon_t^L \) can capture the both conventional and unconventional monetary policies.

In the non-ZLB environment, \( s_t = r_t > 0 \) , \( b_t^L = b_t^G \) and \( r_t^p = r_t + r^p + \epsilon_t^L \), the New Keynesian IS curve is

\[ x_t = E_t x_{t+1} - \frac{1}{\sigma} (r_t^B - E_t \pi_{t+1} - r_t^N) \]

\[ = E_t x_{t+1} - \frac{1}{\sigma} (r_t + r^p + \epsilon_t^L - E_t \pi_{t+1} - r_t^N) \]

\[ = E_t x_{t+1} - \frac{1}{\sigma} (s_t + E_t \pi_{t+1}) + \epsilon_t^L \]  \hspace{1cm} (3)

where \( \epsilon_t^L = -\frac{1}{\sigma} (r^p + \epsilon_t^L - r_t^N) \) is a compound of exogenous shocks. The risk premium shock \( r_t^L \) and \( r_t^N \) can’t be identified separately, so we denote the compound of these exogenous shocks as a demand shock \( \epsilon_t^L \).

In the ZLB environment, the New Keynesian IS curve is

\[ x_t = E_t x_{t+1} - \frac{1}{\sigma} (r_t^B - E_t \pi_{t+1} - r_t^N) \]

\[ = E_t x_{t+1} - \frac{1}{\sigma} (s_t + r^p + \epsilon_t^L - E_t \pi_{t+1} - r_t^N) \]

\[ = E_t x_{t+1} - \frac{1}{\sigma} (s_t - E_t \pi_{t+1}) + \epsilon_t^L \]  \hspace{1cm} (4)

which is same as its counterpart in the non-ZLB environment. We can find that from equation (3) and (4), New Keynesian IS curve with shadow rate has the same specification in both ZLB environment and non-ZLB environment.

Finally, we define a Taylor rule of shadow rate with interest rate smoothing.

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Figure 5: Credit Spread and Balance Sheet
\[ s_t = \varphi_s s_{t-1} + \frac{1 - \varphi_s}{\sigma} (\xi_t \pi_t + \varphi_s \pi_t) + \epsilon_t^\pi \] (5)

where \( \epsilon_t^\pi \) is the monetary policy shock and \( \varphi_s > 1 \) guarantees a unique, non-explosive equilibrium.

2.3. Estimation of NK-DSGE Model with Shadow Rate

From the analysis in Section 3.2, we can find the NK-DSGE model with shadow rate has the same formulation in both ZLB and non-ZLB environment given equation (3) and equation (4). Because we have three observable variables, output gap, inflation rate and shadow rate, we add a shock term to the New Keynesian Phillips Curve (equation (2)) to avoid the stochastic singularity in estimation. According to Gali (2015, chapter 4), the shock term can be explained as a cost-push shock which may come from the exogenous variations in desired price markups or exogenous variations in wage markups. The main model is given in equation (6), (7), (8) and (9).

\[ x_t = \frac{1}{\sigma} (s_t - \varphi_s \pi_t) + \epsilon_t^\pi \] (6)

\[ \pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \epsilon_t^\pi \] (7)

\[ s_t = \varphi_s s_{t-1} + (1 - \varphi_s)(\xi_t \pi_t + \varphi_s \pi_t) + \epsilon_t^\pi \] (8)

All shocks follow first-order autoregressive processes

\[ \epsilon_t^\text{shock} = \mu^\text{shock} + \mu^\text{shock} \epsilon_{t-1} + \epsilon_t^\pi \] (9)

where \( \mu^\text{shock} \sim N(0, \sigma^\text{shock}) \) is exogenous innovation term.

Figure 6: Data for Estimation

The data used for output gap \( x_t \) in equation (6) which is official estimate obtained from BoJ is from 1983Q1 to 2016Q3. The data series of inflation \( \pi_t \) in equation (7) is GDP deflator-based inflation rate from 1980Q3 to 2016Q3. The data series of shadow rate in equation (8) is from 1999Q1 to 2016Q3. For non-ZLB period from 1999Q1 to 1999Q4, the shadow rate is replaced by the non-ZLB constrained call rate to complete the full data series of interest rate. Some structural parameters are calibrated as footnote as \( \alpha = 0 \), \( \beta = 0.9975 \), \( \epsilon = 6 \) and \( \sigma = \eta = 1 \). The elasticity of substitution \( \epsilon \) is calibrated to be 6 which means an average 20% markup charged by intermediate-good firms at steady state. \( \sigma \) and \( \eta \) are calibrated at 1 because these parameters can’t be identified in the Bayesian estimation. \( \alpha = 0 \) means the model economy has the constant scale to return. \( \kappa \) is a composite of other structural parameters and we specify its prior distribution as non-informative uniform prior distribution U(0,1). The prior and posterior distributions of parameters and standard deviations are given in Table 3. We estimate the NK-DSGE in a DSGE-VAR style to compare the theoretical impulse response from DSGE model and corresponding empirical impulse response from Bayesian VAR model. Please refer to Del Negro and Shorfheide (2004), Adjemian et al. (2008), Del negro et al. (2007) for the technical details of DSGE-VAR model. The basic idea of the DSGE-VAR(\( \lambda \)) is to use the empirical moments implied from a structural-from DSGE model as the prior distribution for a Bayesian VAR model. When choosing the prior distribution of a reduced-form Bayesian VAR model, \( \lambda \) is the weight of this constraint of empirical moments implied from the DSGE model. Following Adjemian et al. (2008), we treat \( \lambda \) as a parameter which can be jointly estimated from the Bayesian estimation of other structural parameters and specify the prior distribution of \( \lambda \) as non-informative uniform prior distribution U(0,2). Also, we compare the impulse response and calculate the historical decomposition to check the contribution of the shadow rate monetary policy shock to output gap and inflation.

3. Shadow Rate NK-DSGE Model in Monetary Policy Analysis

By introducing the concept of shadow rate into a standard NK-DSGE model and estimating it with the shadow rate, the monetary policy analysis in the ZLB environment can be done in a similar way as the analysis in the non-ZLB environment. We run some standard procedures of DSGE methodology. Firstly, we show the historical decomposition to check the contributions of each shock.

3.1. Historical Decomposition

From the Figure 7, we can confirm the effects of monetary policy represented by shadow rate, especially from 2014.
Table 3: Bayesian Estimation of Structural Parameters  

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Prior Distribution</th>
<th>Mean</th>
<th>St.Dev.</th>
<th>Prior</th>
<th>Mean</th>
<th>90% HPD</th>
<th>Mean</th>
<th>St.Dev.</th>
<th>Prior</th>
<th>Mean</th>
<th>90% HPD</th>
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<td>$\varphi_{\pi}$</td>
<td>1.5 0.1 G</td>
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<td>1.4679</td>
<td>0.1035</td>
<td>[1.2925,1.6328]</td>
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</tr>
<tr>
<td>$\varphi_{\chi}$</td>
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<td>0.1434</td>
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<td>$\varphi_{S}$</td>
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<td>0.0341</td>
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<td>0.8506</td>
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<tr>
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<td>0.4475</td>
<td>0.0906</td>
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<tr>
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<td>0.7713</td>
<td>0.0800</td>
<td>[0.6460,0.9060]</td>
<td>0.7949</td>
<td>0.0658</td>
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<td>[0.1088,0.1629]</td>
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Parameters $\kappa$ and $\lambda$ follow $U(0,1)$ and $U(0,2)$ distributions, respectively. 

The positive contributions of the shadow rate monetary policy shock (red area in Figure 7) from 2014 confirm the effects of Quantitative Qualitative Monetary Easing (QQE) of BoJ. 

Figure 7: Historical Decomposition of Inflation

Figure 8 shows the historical decompositions of inflation. The most of fluctuations of inflation come from the supply shock, demand shock and (shadow rate) monetary policy shock. Also, (shadow rate) monetary policy shock has positive contribution to positive inflation since 2014 (red area in Figure 8). We can conclude from the historical decompositions of output gap and inflation, the enlargement of balance sheet of BoJ does have its effect to improve the output gap and deflation. 

3.2. Impulse Response Function

Bayesian impulse response functions (IRFs) are calculated based on the posterior distributions of structural parameters and exogenous shock standard deviations. The gray shaded area represents the 90% Highest Posterior Density Interval (HPDI) of posterior impulse response calculated from DSGE model. The thick black line is the medium of posterior impulse response of DSGE model. The corresponding impulse response calculated from Bayesian VAR is also plotted in dashed lines with HPDI (90%) and the medium of impulse response inside the two dashed lines. We can find the posterior impulse response from DSGE model and VAR model has the almost same dynamics, similar range and direction, especially for inflation and (shadow) interest rate. Note that even the shadow rate can take negative values, the change of shadow rate, a rise or a reduction, still has the same effect as the positive policy rate has in the non-ZLB environment. The shadow rate NK-DSGE model shows consistent IRFs no matter the policy rate is restricted by the ZLB or not, both models show the output gap improves due to the expansionary monetary policy shock. 

Figure 9: IRF of negative (shadow rate) monetary policy shock $\varepsilon_{t}^{s}$

Figure 10 shows the dynamic response of demand shock $\varepsilon_{t}^{x}$. A positive demand shock can improve the output gap. At the same time, monetary policy rule responses to the corresponding increasing of inflation and output gap with the increasing of (shadow) interest rate. 

Figure 10: IRF of demand shock $\varepsilon_{t}^{x}$
Figure 11: IRF of supply shock $\varepsilon^\pi_t$

We find that all these empirical results are same to those in the standard NK-DSE model although here we used the shadow rate to estimate the model.

DISCUSSIONS

In this paper, we estimated the shadow rate of Japan economy from a shadow rate term structure model. We adopt the shadow rate as a measure of monetary policy of BoJ because since 1999Q1, the general policy rate of BoJ, call rate, has kept been near zero and already lost its function as an operating target for the conduct of monetary policy. The concept of shadow rate is not new and for a long time, it is not widely used in macroeconomics. Since the ZLB has become a common issue for monetary authorities in the advanced economies, using the shadow rate to observe and analyze the monetary policy has been applied in many empirical works. We also confirmed that the shadow rate does have credible traceability of the BoJ's policy in the ZLB environment, providing us a new perspective to check the unconventional monetary policy of BoJ. Most of existing literatures use the shadow rate in reduced-form time series econometric models such as FAVAR or TVP-VAR to find the empirical evidences of unconventional monetary policy, but these models are not structural. The introduction of the shadow rate into the DSGE framework is a new attempt. Using the shadow rate does relieve us from the technical difficulties incurred by the ZLB, but whether this method is robust or not is still unclear. As far as we know, there doesn't exist other similar works that use the shadow rate in the estimation of DSGE model. However, the historical decomposition of output gap and inflation advocates the effectiveness of the monetary easing conducted by BoJ. The transmission channels of the monetary easing policy which are assumed in this paper are based on the empirical findings and these empirical evidences have been mapped into the model. Then we find that the impulse response functions of shadow rate NK-DSE model are similar to the benchmark NK-DSE model. The mechanism of the general policy rate and shadow rate is same as the standard NK-DSE framework.

Note that there may exist one defect in the estimation of DSGE models with shadow rate. It is that the shadow rate is not endogenously derived from a structural model, but exogenously estimated by a statistical model. SRTS model is a factor model and the factors are used to fit and describe the yield curve in a ZLB environment. But the factors have less economic interpretation. Krippner (2015) gives a structural interpretation of these factors in a linear economy framework, but this framework is highly stylized and has less dynamics than DSGE models. The estimation of the shadow rate by SRTS model needs nonlinear numerical calculations, this makes the endogenous determination of the shadow rate in a structural DSGE framework very difficult.

For further research, we want to introduce the shadow rate to a Smets and Wouters (2003, 2007) type middle-scale DSGE model, which is the prototype of the DSGE models used in major central banks. Also, we shouldn't forget that the shadow rate only exists as an economic concept and the central bank can't control the shadow rate as an operating target, but we can use the information from it as a guidance for monetary policy operation.

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